

# On the index of equivariant Toeplitz operators

Ulrich Bunke\*

February 1, 2008

## Contents

<b>1</b>	<b>Introduction</b>	<b>0</b>
<b>2</b>	<b>The index of Toeplitz operators</b>	<b>0</b>
<b>3</b>	<b><math>\Gamma</math>-equivariant Toeplitz operators</b>	<b>3</b>
<b>4</b>	<b>Examples</b>	<b>7</b>

## 1 Introduction

The goal of the present paper is to understand aspects of the recent preprint [6]. While this paper studies traces of commutators the present paper concentrates on the index theoretic aspects. This allows for studying the index Toeplitz operators under quite general assumptions. The basic observation (compare [5]) is that the index of the Toeplitz operator is equal to the index of an associated Callias type operator, i.e. a Dirac operator with potential. Callias type operators were thoroughly studied in [3]. In particular, their index is very accessible to computation. In the present note we show how to extend all that to the equivariant case.

## 2 The index of Toeplitz operators

In this Section we review the non-equivariant situation. The main ideas can be traced back to [5].

---

\*Mathematisches Institut, Universität Göttingen, Bunsenstr. 3-5, 37073 Göttingen, GERMANY, E-mail: bunke@uni-math.gwdg.de

Let  $(M, g)$  be a complete Riemannian manifold and  $E \rightarrow M$  be a Dirac bundle which may be  $\mathbf{Z}_2$ -graded or ungraded. The associated Dirac operator is an unbounded essentially selfadjoint operator on the Hilbert space  $H := L^2(M, E)$  with domain  $C_c^\infty(M, E)$ .

**Assumption 2.1** *We assume that 0 is an isolated point of the spectrum of  $D$ .*

Let  $\mathcal{H}$  denote the kernel of  $D$ , and let  $P$  be the orthogonal projection onto  $\mathcal{H}$ . In the graded case we have an orthogonal splitting  $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ ,  $P = P^+ + P^-$ .

Let  $C_g^\infty(M)$  denote the space of all bounded smooth functions such that  $df$  vanishes at infinity of  $M$ . The commutative  $C^*$ -algebra  $C_g(M)$  is defined as the closure of  $C_g^\infty(M)$  inside  $C(M)$ . Furthermore, let  $C_0(M)$  denote the closure of  $C_c^\infty(M)$  in  $C(M)$ .

If  $f \in C(M)$ , then  $M_f$  denotes the multiplication operator on  $H$  induced by  $f$ .

**Definition 2.2** *We define the following operators on  $\mathcal{H}$ , resp.  $\mathcal{H}^\pm$ :*

$$T_f := PM_fP, \quad T_f^\pm := P^\pm M_f P^\pm.$$

We extend this definition to matrix valued functions  $f \in \text{Mat}(n, C(M))$  such that  $M_f \in B(H \otimes \mathbf{C}^n)$ ,  $T_f \in B(\mathcal{H} \otimes \mathbf{C}^n)$ , etc.

**Lemma 2.3** *If  $f \in C_0(M)$ , then  $T_f$  is compact.*

*Proof.* If  $f \in C_c^\infty(M)$ , then  $M_fP$  is compact by Rellich's Lemma since  $\mathcal{H} \subset H^1$ , where the latter space is the Sobolev space defined as the domain of the closure of the elliptic operator  $D$ . The map  $f \mapsto M_f$  is continuous. Since the space of compact operators is norm-closed we conclude that  $M_fP$  is compact for all  $f \in C_0(M)$ .  $\square$

**Lemma 2.4** *If  $f \in C_g(M)$ , then  $[P, M_f]$  is compact.*

*Proof.* It is here where we use the assumption that 0 is an isolated point of  $\sigma(D)$ . Let  $B \subset \mathbf{C}$  be a small ball around zero such that  $\sigma(D) \cap B = \{0\}$ . For  $\lambda \notin \sigma(D)$  let  $R_D(\lambda) := (\lambda - D)^{-1}$  denote the resolvent. By holomorphic function calculus we can write

$$P = \frac{1}{2\pi i} \int_{\partial B} R_D(\lambda) d\lambda.$$

Let  $f \in C_g^\infty(M)$ . Then we have

$$\begin{aligned} [R_D(\lambda), M_f] &= R_D(\lambda)[D, M_f]R_D(\lambda) \\ &= R_D(\lambda)c(\text{grad} f)R_D(\lambda), \end{aligned}$$

where  $c$  denotes Clifford multiplication. Since  $\text{grad} f$  vanishes at infinity we conclude that  $c(\text{grad} f)R_D(\lambda)$  is compact. In fact, we can approximate  $\text{grad} f$  uniformly by vector fields  $X$  of compact support, and  $XR_D(\lambda)$  is again compact by Rellich's Lemma since  $R_D(\lambda)$  maps  $H$  continuously to  $H^1$ . Thus  $[R_D(\lambda), M_f]$  is compact, too. Again referring to the norm-closedness of the space of compact operators on  $H$  we see that

$$[P, M_f] = \frac{1}{2\pi i} \int_{\partial B} [R_D(\lambda), M_f] d\lambda$$

is compact for all  $f \in C_g^\infty(M)$ , and hence for all  $f \in C_g(M)$ .  $\square$

Note that Lemma 2.3 and 2.4 extend to matrix valued functions.

The underlying topological space of  $M$  can be considered as the spectrum of the commutative  $C^*$ -algebra  $C_0(M)$ . The exact sequence of  $C^*$ -algebras

$$0 \rightarrow C_0(M) \rightarrow C_g(M) \rightarrow C(\partial_g M) \rightarrow 0$$

defines the compactification of  $M$  by the Higson corona  $\partial_g M$ , where  $\partial_g M$  is the spectrum of the commutative  $C^*$ -algebra  $C(\partial_g M) := C_g(M)/C_0(M)$ .

We now consider a continuous unitary matrix valued function  $F : \partial_g M \rightarrow U(n)$ , or equivalently, a unitary  $F \in U(n, C(\partial_g M))$ . In the ungraded case we assume in addition that  $F$  is selfadjoint. Let  $f, g \in \text{Mat}(n, C_g(M))$  be extensions of  $F$  and  $F^{-1}$  to  $M$ .

**Lemma 2.5** *The operator  $T_f$  is a Fredholm operator. Its index only depends on  $F$ , where we define the index in the graded case as  $\text{index}(T_f) := \text{index}(T_f^+) - \text{index}(T_f^-)$ .*

*Proof.* A parametrix of  $T_f$  is given by  $T_g$ . In fact, if " $\sim$ " denotes equality modulo compact operators, then we have by Lemma 2.4 that  $T_f T_g \sim T_{fg} = 1 + T_{fg-1}$ . Since  $fg - 1 \in \text{Mat}(n, C_0(M))$  we have by Lemma 2.3 that  $T_{fg-1} \sim 0$ . Similarly we show that  $T_g T_f \sim 1$ . If  $f, f'$  are two extensions of  $F$ , then  $f - f' \in \text{Mat}(n, C_0(M))$ . By Lemma 2.3 we conclude that  $T_f \sim T_{f'}$  and hence equality of the indices.  $\square$

The goal of the present section is to relate the index of  $T_f$  with the index of the Callias-type operator constructed in [3] from  $D$  and  $f, g$ . The Theorems 2.9 and 2.16 of [3] reduce the computation of the index of the Callias type operator and thus of  $T_f$  to an application of the Atiyah-Singer Index theorem for elliptic differential operators on closed manifolds. In the ungraded case [3], Prop. 2.8 implies that  $\text{index}(T_f) = 0$ , since we assume that  $\sigma(D)$  has a gap.

In the ungraded case we define the Callias-type operator  $C := D + iM_f$  on  $E \otimes \mathbf{C}^n$  and put  $\epsilon := 1$ . In the graded case we define  $\epsilon := -1$  and  $C := D + \text{diag}(-M_g^+, M_f^-)$  on  $E^+ \otimes \mathbf{C}^n \oplus E^- \otimes \mathbf{C}^n$ .

As shown in Sec. 2 of [3] these Callias-type operators have a well-defined index

$$\text{index}(C) := \dim \ker_{L^2}(C) - \dim \ker_{L^2}(C^*) .$$

Our main result is

**Proposition 2.6**

$$\text{index}(T_f) = \epsilon \text{index}(C)$$

*Proof.* We first consider the graded case. Let  $H^1$  be the domain of the closure of (the extension of)  $D$  (from  $E$  to  $E \otimes \mathbf{C}^n$ ) and put  $H := L^2(M, E \otimes \mathbf{C}^n)$ . Then  $C : H^1 \mapsto H$  is Fredholm in the usual sense. Set  $\Phi := \text{diag}(-M_g, M_f)$  and  $Q := 1 - P$ .  $P$  and  $Q$  act on  $H$  as well as on  $H^1$ . It follows from Lemma 2.5 that  $Q\Phi P$  and  $P\Phi Q$  are compact operators from  $H^1$  to  $H$ . Therefore we can replace  $C$  by  $PCP \oplus QCQ$  without changing the index. Using the homotopy  $QC_tQ$ ,  $C_t := D + t\Phi$  we can deform  $QCQ$  to the invertible operator  $QC_0Q$  through Fredholm operators. Here we use the fact that  $C_t$  is Fredholm for all  $t > 0$ . We conclude that

$$\begin{aligned} \text{index}(C) &= \text{index}(PCP) = \text{index}(P\Phi P) \\ &= \text{index}(T_g^+) + \text{index}(T_f^-) = -(\text{index}(T_f^+) - \text{index}(T_f^-)) \\ &= -\text{index}(T_f) . \end{aligned}$$

We now come to the ungraded case. We again have

$$\text{index}(C) = \text{index}(PCP) + \text{index}(QCQ) = \text{index}(PCP) = \text{index}(T_f) ,$$

since  $QCQ$  can be deformed to the invertible operator  $QC_0Q$ , where  $C_t := D + itM_f$ .  $\square$

As explained above we conclude with [3], Prop. 2.8 that

**Corollary 2.7** *In the ungraded case we have  $\text{index}(T_f) = 0$ .*

### 3 $\Gamma$ -equivariant Toeplitz operators

Let  $\tilde{M} \rightarrow M$  be a Galois cover with group of deck transformations  $\Gamma$ . We reserve the symbol “ $\sim$ ” to denote lifts of various objects to  $\Gamma$ -coverings.

**Assumption 3.1** *We can choose a cut-off function  $\chi^\Gamma \in C_g^\infty(\tilde{M})$  such that  $\sum_{\gamma \in \Gamma} \gamma^* \chi^\Gamma = 1$ ,  $\#\{\gamma \in \Gamma \mid \gamma^* \chi^\Gamma \chi^\Gamma \neq 0\} < \infty$ .*

In our motivating example  $\tilde{M}$  is a symmetric space of rank one such that  $\Gamma$  is a convex-cocompact group of isometries. In this case  $\chi^\Gamma$  exists by [4], Lemma 6.4.

**Assumption 3.2** *We assume that 0 is an isolated point of the spectrum of  $\tilde{D}$ .*

If  $\tilde{D}$  is a homogeneous Dirac operator on a symmetric space  $\tilde{M}$  of rank one and  $M$  is a quotient by a convex cocompact subgroup, then by the result of [4] we have  $\sigma_{ess}(D) = \sigma_{ess}(\tilde{D})$ . In particular, if  $\tilde{D}$  is one of the Dirac operators constructed by [1] in order to realize the representations of the discrete series, then the assumptions 2.1 and 3.2 are satisfied.

We consider the Hilbert space  $\tilde{H} := L^2(\tilde{M}, \tilde{E})$  which carries an unitary representation of  $\Gamma$ . Let  $B_\Gamma := {}^\Gamma B(\tilde{H})$  denote the  $\Gamma$ -equivariant bounded operators on  $\tilde{H}$ . Using a fundamental domain  $F \subset \tilde{M}$  we can write  $\tilde{H} = L^2(\Gamma) \otimes H$  in a  $\Gamma$ -equivariant way. Let  $\mathcal{N}(\Gamma) \subset B(L^2(\Gamma))$  be the group von Neumann algebra of all operators commuting with left translations. Then  $B_\Gamma = \mathcal{N}(\Gamma) \otimes B(H)$ . Let  $K_\Gamma \subset B_\Gamma$  be the ideal of  $\Gamma$ -compact operators corresponding to  $\mathcal{N}(\Gamma) \otimes K(H)$ . An operator  $A \in B_\Gamma$  is called  $\Gamma$ -Fredholm if it is invertible modulo  $K_\Gamma$ . If  $A$  is  $\Gamma$ -Fredholm, then its index is an element  $\text{index}(A) \in K_0(K_\Gamma) = K_0(\mathcal{N}(\Gamma))$ . The normalized trace  $\tau$  on the  $II_1$ -factor  $\mathcal{N}(\Gamma)$  induces a homomorphism  $\tau : K_0(\mathcal{N}(\Gamma)) \rightarrow \mathbf{R}$ . We define the  $\Gamma$ -index of  $A$  by  $\text{index}_\Gamma(A) := \tau(\text{index}(A))$ .

**Lemma 3.3** *If  $A \in B_\Gamma$  and  $M_{\chi^\Gamma} A$  is compact, then  $A$  is  $\Gamma$ -compact.*

*Proof.* We make the isomorphism  $I : B_\Gamma \xrightarrow{\sim} \mathcal{N}(\Gamma) \otimes B(H)$  explicit. First we identify  $H = L^2(F, \tilde{E}|_F)$ . Then  $i : \tilde{H} \xrightarrow{\sim} L^2(\Gamma) \otimes L^2(F, \tilde{E}|_F)$  is given by  $i(\phi) := \sum_{\gamma \in \Gamma} \delta_\gamma \otimes \chi_F \gamma^{-1} \phi$ , where  $\delta_\gamma(\gamma')$  is zero for  $\gamma \neq \gamma'$  and 1 in the remaining case, and  $\chi_F$  denotes the characteristic function of  $F$ . The inverse of this identification is given by  $i^{-1}(\sum_{\gamma \in \Gamma} \delta_\gamma \otimes \phi_\gamma) = \sum_{\gamma \in \Gamma} \gamma \phi_\gamma$ . We now compute  $I(A) = i \circ A \circ i^{-1}$

$$\begin{aligned} I(A) \left( \sum_{\gamma \in \Gamma} \delta_\gamma \otimes \phi_\gamma \right) &= \sum_{\gamma, \gamma' \in \Gamma} \delta_{\gamma'} \otimes \chi_F(\gamma')^{-1} \gamma A \phi_\gamma \\ &= \sum_{\gamma, \gamma' \in \Gamma} \delta_{\gamma \gamma'} \otimes \chi_F(\gamma')^{-1} A \phi_\gamma \\ &= \left( \sum_{\gamma' \in \Gamma} R(\gamma') \otimes A_{\gamma'} \right) \left( \sum_{\gamma \in \Gamma} \delta_\gamma \otimes \phi_\gamma \right), \end{aligned}$$

thus  $I(A) = \sum_{\gamma \in \Gamma} R(\gamma) \otimes A_\gamma$ , where  $A_\gamma := \chi_F \gamma^{-1} A \chi_F$  and  $R(\gamma) \in \mathcal{N}(\gamma)$  is the right translation by  $\gamma$ . If  $M_{\chi^\Gamma} A$  is compact, then so is  $M_{\gamma^* \chi^\Gamma} A$  for all  $\gamma \in \Gamma$ . For all  $\gamma \in \Gamma$  we see that  $\chi_F \gamma^{-1} A \chi_F = \sum_{\gamma' \in \Gamma} \chi_F \gamma^{-1} M_{(\gamma')^* \chi^\Gamma} A \chi_F$  is compact since  $\chi_F \gamma^{-1} M_{(\gamma')^* \chi^\Gamma} \neq 0$  for at most finitely many  $\gamma' \in \Gamma$ . Thus  $A$  is  $\Gamma$ -compact.  $\square$

If  $f \in C_g(M)$ , then we have the multiplication operator  $M_{\tilde{f}} \in {}^\Gamma B_\Gamma$ . Let  $\tilde{\mathcal{H}}$  denote the kernel of  $\tilde{D}$  and  $\tilde{P} \in {}^\Gamma B(\tilde{H})$  the orthogonal projection onto  $\tilde{\mathcal{H}}$ .

**Definition 3.4** We define  $\tilde{T}_f := \tilde{P}M_{\tilde{f}}\tilde{P} \in B_\Gamma$ .

**Lemma 3.5** If  $f \in C_0(M)$ , then  $\tilde{T}_f$  is  $\Gamma$ -compact.

*Proof.* We have

$$M_{\chi^\Gamma} \tilde{T}_f = M_{\chi^\Gamma} \tilde{P} M_{\tilde{f}} \tilde{P} \stackrel{\text{Lemma 2.4}}{\sim} \tilde{P} M_{\chi^\Gamma \tilde{f}} \stackrel{\text{Lemma 2.3}}{\sim} 0$$

since  $\chi^\Gamma \tilde{f} \in C_0(\tilde{M})$ . The assertion now follows from Lemma 3.3.  $\square$

**Lemma 3.6** If  $f \in C_g(M)$ , then  $[\tilde{P}, M_{\tilde{f}}]$  is  $\Gamma$ -compact.

*Proof.* We employ the same method as in the proof of Lemma 2.4 using

$$M_{\chi^\Gamma} \tilde{R}(\lambda) c(\text{grad} \tilde{f}) \tilde{R}(\lambda) \sim \tilde{R}(\lambda) M_{\chi^\Gamma} c(\text{grad} \tilde{f}) \tilde{R}(\lambda) \sim 0$$

since  $M_{\chi^\Gamma} c(\text{grad} \tilde{f})$  vanishes at infinity of  $\tilde{M}$ . The assertion now follows from Lemma 3.3.  $\square$

Consider  $F \in U(n, C(\partial_g M))$  which is selfadjoint in the ungraded case. Let  $f, g \in \text{Mat}(n, C_g(M))$  be lifts of  $F, F^{-1}$ .

**Lemma 3.7** The operator  $\tilde{T}_f$  is  $\Gamma$ -Fredholm. Its index only depends on  $F$ .

*Proof.* The proof is analogous to the proof of Lemma 2.5. One has to replace "compact" by " $\Gamma$ -compact" and applies Lemma 3.5 and 3.6 instead of Lemma 2.3 and 2.4.  $\square$

The following theorem is the main result of the present paper.

**Theorem 3.8**

$$\text{index}_\Gamma(\tilde{T}_f) = \text{index}(T_f) .$$

*Proof.* The lifts  $\tilde{f}, \tilde{g}$  give rise to a  $\Gamma$ -equivariant Callias type operator  $\tilde{C}$ . In the first step we show that  $\tilde{C}$  is  $\Gamma$ -Fredholm, and that its index coincides with the index of  $\tilde{T}_f$  (up to the sign  $\epsilon$ ). In [3], Sec.2 the computation of the index of  $C$  was reduced to the computation of the index of an elliptic differential operator  $R$  on a closed manifold  $S^1 \times N$  using a relative index theorem

and a cut-and-past procedure. Doing this cut-and-paste procedure equivariantly in the second step we reduce the computation of the index of  $\tilde{C}$  to the computation of the index of the lift  $\tilde{R}$  of  $R$  to a certain cover  $S^1 \times \tilde{N}$ . In the third and final step we apply the Atiyah's index theorem for coverings in order to conclude that the  $\Gamma$ -index of  $\tilde{R}$  coincides with the index of  $R$ .

We form the flat bundle of von Neumann algebras  $\mathcal{V} := \tilde{M} \times_{\Gamma} \mathcal{N}(\Gamma)$  and let  $D_{\mathcal{V}}$  be the  $\mathcal{N}(\Gamma)$ -equivariant twisted Dirac operator on  $E \otimes \mathcal{N}(\Gamma)$ . We now form the  $\mathcal{N}(\Gamma)$ -equivariant Callias type operators  $\hat{C} := D_{\mathcal{V}} + iM_f$  on  $E \otimes \mathcal{V} \otimes \mathbf{C}^n$  in the ungraded and  $\hat{C} := D_{\mathcal{V}} + \text{diag}(-M_g^+, M_f^-)$  on  $E^+ \otimes \mathcal{V} \otimes \mathbf{C}^n \oplus E^- \otimes \mathcal{V} \otimes \mathbf{C}^n$  in the graded case. Combining [3], Lemma 2.6, 2.14, and the proof of Lemma 1.18 we show that the operator  $\hat{C}$  is invertible at infinity (see [3], Ass. 1). It follows that  $\hat{C}$  induces a Fredholm operator between the Hilbert- $\mathcal{N}(\Gamma)$  modules  $H^1(M, E \otimes \mathcal{V} \otimes \mathbf{C}^n)$  and  $L^2(M, E \otimes \mathcal{V} \otimes \mathbf{C}^n)$ . The tensor products over  $\mathcal{N}(\Gamma)$  of these modules with  $L^2(\Gamma)$  identify with  $H^1(\tilde{M}, \tilde{E} \otimes \mathbf{C}^n)$  and  $L^2(\tilde{M}, \tilde{E} \otimes \mathbf{C}^n)$ , respectively. The operator  $\hat{C}$  gives rise to the  $\Gamma$ -equivariant Callias type operator  $\tilde{C}$  which is just the lift of  $C$ . In particular, we see that  $\tilde{C}$  is  $\Gamma$ -Fredholm and  $\text{index}(\hat{C}) = \text{index}(\tilde{C})$ . We can now apply exactly the same argument as in the proof of Proposition 2.6 in order to show that  $\text{index}(\tilde{C}) = \epsilon \text{index}(\tilde{T}_f)$ , replacing compactness and Fredholm by the corresponding  $\Gamma$ -equivariant notions. This ends the first step of the proof.

We now come to the second step. In the ungraded case we can repeat the argument of the proof of [3], Prop. 2.8 in order to see that  $\text{index}(\hat{C}) = 0$  since by assumption there is a gap in the spectrum of  $D_{\mathcal{V}}$  (note that the spectrum of  $D_{\mathcal{V}}$  coincides with that of  $\tilde{D}$ ). Thus  $0 = \text{index}_{\Gamma}(\tilde{T}_f)$ . Since  $\text{index}(T_f) = 0$  by Corollary 2.7 we obtain the assertion on the theorem in the ungraded case. It remains to consider the graded case. Doing the construction [3], 2.4.2 with  $D$  and  $D_{\mathcal{V}}$  at the same time we arrive at a Dirac operator  $R$  and its twist  $R_{\mathcal{W}}$  over a compact manifold  $S^1 \times N$  such that  $\text{index}(C) = \text{index}(R)$  and  $\text{index}(\hat{C}) = \text{index}(R_{\mathcal{W}})$ . Here  $N$  is a certain closed hypersurface of  $M$ ,  $R$  is associated to a Dirac bundle  $L \mapsto S^1 \times N$ , and  $\mathcal{W}$  is the pull-back to  $S^1 \times N$  of the restriction of  $\mathcal{V}$  to  $N$ . Let  $\tilde{N}$  be the restriction of the cover  $\tilde{M} \rightarrow M$  to  $N$ . The tensor products over  $\mathcal{N}(\Gamma)$  of  $H^1(S^1 \times N, L \otimes \mathcal{W})$ ,  $L^2(S^1 \times N, L \otimes \mathcal{W})$  with  $L^2(\Gamma)$  identify with  $H^1(S^1 \times \tilde{N}, \tilde{L})$  and  $L^2(S^1 \times \tilde{N}, \tilde{L})$ , respectively. The operator  $R_{\mathcal{W}}$  induces the  $\Gamma$ -equivariant Dirac operator  $\tilde{R}$  on  $\tilde{L}$  which is just the lift of  $R$ . We have  $\text{index}(\tilde{R}) = \text{index}(R_{\mathcal{W}})$ . This accomplishes the second step.

In the last step we apply Atiyah's index theorem for coverings [2] in order to conclude that  $\text{index}_{\Gamma}(\tilde{R}) = \text{index}(R)$ . This implies  $\text{index}_{\Gamma}(\tilde{T}_f) = \text{index}(T_f)$  by Proposition 2.6 and the first two steps.  $\square$

## 4 Examples

We first consider the two-dimensional example. Let  $\tilde{M}$  be the hyperbolic plane and  $\Gamma$  be a convex cocompact subgroup of the group of isometries of  $\tilde{M}$ . The geodesic boundary  $\partial\tilde{M}$  can be decomposed into a limit set  $\Lambda$  and its complement  $\Omega$ . The group  $\Gamma$  acts freely and properly on  $\tilde{M} \cup \Omega$ , and the compact manifold with boundary  $\bar{M} := \Gamma \backslash \tilde{M} \cup \Omega$  is the geodesic compactification of  $M := \Gamma \backslash \tilde{M}$ . The boundary  $B := \partial\bar{M}$  is a finite union of circles  $\Gamma \backslash \Omega$ . There is a natural projection of the Higson corona  $\partial_g M$  to  $B$ . Thus any  $U(1)$ -valued function  $F$  on  $B$  can be lifted to  $\partial_g M$ , and we will denote this lift by the same symbol.

Note that  $M$  is a complex manifold. Let  $K$  be the canonical bundle of  $M$ . We fix  $k \in \mathbf{Z}$  and consider the graded Dirac operator  $D = \bar{\partial} + (\bar{\partial})^*$  on  $E = K^k \oplus K^{k-1}$ , where  $\bar{\partial} : C^\infty(M, K^k) \rightarrow C^\infty(M, K^{k-1})$  is the Dolbeault operator.

The complex structure fixes an orientation of  $M$  which induces an orientation of  $B$ . We now use the notation of [3], 2.16. The Dirac operator  $D_N$  is just  $i \frac{\partial}{\partial t}$  on any component of  $B$ , where  $t$  is the coordinate of  $S^1$  compatible with the orientation. The index  $\text{index}(T_f)$  is minus the spectral flow of the family connecting  $D_N$  and  $F^* D_N F$ , and this is equal to the total winding number  $n(F)$  of  $F$ , i.e.

$$\text{index}(T_f) = -\frac{1}{2\pi i} \int_B F^{-1} dF .$$

Let  $P_k$  denote the projection onto the space of holomorphic square integrable sections of  $\tilde{K}^k$ . Note that  $P_0 = 0$ . For  $k > 0$  ( $k < 0$ ) the range of  $P_k$  is the holomorphic (antiholomorphic) discrete series representation of  $PSL(2, R)$ , the orientation-preserving isometry group of  $\tilde{M}$ . Let  $\tilde{T}_f^k := P_k \tilde{f} P_k$  be the Toeplitz compressions. Then by the computation above and Theorem 3.8 for  $k \neq 0$  we have  $\text{index}_\Gamma(\tilde{T}_f^k) = -\text{sign}(k)n(F)$ .

This has the following higher-dimensional generalization. Let  $\tilde{M} := Spin(1, 2n)/Spin(2n)$  be the real hyperbolic space of dimension  $2n$  and  $\Gamma \subset Spin(1, 2n)$  be a convex cocompact subgroup. Let  $\tilde{S}$  be the spinor bundle of  $\tilde{M}$  and  $\tilde{V}$  be any further  $Spin(1, 2n)$ -homogeneous bundle. We put  $\tilde{E} := \tilde{S} \otimes \tilde{V}$  and let  $\tilde{D}$  be the associated Dirac operator. We assume that  $\tilde{V}$  is such that 0 is an isolated point of the spectrum of  $\tilde{D}$ . In this case  $\ker \tilde{D}$  decomposes into a finite sum of discrete series representations of  $Spin(1, 2n)$ .

We again have a decomposition of the geodesic boundary of  $\tilde{M}$  into a limit set and a domain of discontinuity  $\Omega$ . The locally symmetric space  $M := \Gamma \backslash \tilde{M}$  can be compactified by adjoining the boundary  $B := \Gamma \backslash \Omega$ . The topology of  $B$  can be quite complicated. Since  $Spin(1, 2n)$  acts on the sphere  $\partial\tilde{M}$  by orientation-preserving conformal transformations  $B$  admits a locally conformally flat structure. In particular, all Pontrjagin classes of  $TB$  and all associated bundles vanish.

Again we have a natural map from the Higson corona of  $M$  to  $B$ . Let  $F : B \rightarrow U(m)$  be a



continuous function and  $f, g \in \text{Mat}(m, C_g(M))$  extensions of  $F, F^{-1}$ . The function  $F$  represents an element  $[F]$  in  $K^1(B)$ . Let  $\mathbf{ch}([F]) \in H^{\text{odd}}(B, \mathbf{R})$  be the Chern character of  $[F]$ . We define the degree of  $F$  by  $\deg(F) := \mathbf{ch}_{2n-1}([F])([B])$ . Here  $[B]$  is the orientation of  $B$  as the boundary of  $M$ , where the orientation of  $M$  is determined by the  $\mathbf{Z}_2$ -grading of  $S$ .

**Proposition 4.1**  $\text{index}_\Gamma(\tilde{T}_f) = \text{index}(T_f) = -\dim(V) \deg(F)$ .

*Proof.* We have to compute the index of the Callias type operator  $C$  on  $M$  which is associated to  $f, g$ . By [3], Thm. 2.16 it is equal to the index of the Dirac operator  $D_L$  on  $S^1 \times B$  twisted with a bundle  $L = L_F \otimes L_V$ . Here  $L_V$  is the restriction of  $V$  to  $B$ . The bundle  $L_F$  is obtained from the trivial bundle  $[0, 1] \times B \times \mathbf{C}^m$  by glueing  $(1, b, v)$  with  $(0, b, F(b)v)$ ,  $b \in B$ ,  $v \in \mathbf{C}^m$ . The bundle  $L_V$  is associated to the tangent bundle of  $B$  and thus has vanishing Chern classes. Further note that  $\hat{\mathbf{A}}(T(S^1 \times N)) = 1$ . The index theorem for twisted Dirac operators thus gives

$$\text{index}(D_L) = \dim(V) \mathbf{ch}(L_F)_{[2n]}([S^1 \times N]) = \dim(V) \mathbf{ch}_{2n-1}([F])([B]) .$$

This finishes the proof of the proposition. □

In the situation above we know that  $\ker D$  is infinite-dimensional by [4]. The proposition above would give an alternative index-theoretic proof of this fact.

## References

- [1] M. Atiyah and W. Schmid. A geometric construction of the discrete series for semisimple Lie groups. *Invent. Math.*, 42(1977), 1–62.
- [2] M. F. Atiyah. Elliptic operators, discrete groups and von Neumann algebras. *Asterisque*, 32(1976), 43–72.
- [3] U. Bunke. A  $\mathbf{K}$ -theoretic relative index theorem and Callias-type operators. *Math. Ann.*, 303(1995), 241–279.
- [4] U. Bunke and M. Olbrich. The spectrum of Kleinian manifolds. To appear in *J. Funct. Anal.*, Preprint available at <http://www.uni-math.gwdg.de/bunke/spzperl.dvi>.
- [5] E. Guenter and N. Higson. A note on Toeplitz operators. *Int. J. Math.*, 7(1996), 501–513.
- [6] R. Nest and F. Radulescu. Index of  $\Gamma$ -equivariant Toeplitz operators. Preprint 1999 : math.OA/9911042.